Journal of Applied Mathematics and Mechanics 72 (2008) 208–213

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

The axisymmetric contact problem for an elastic layer loaded with a deformable cover plate☆

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ARTICLE INFO

Article history:
Received 20 February 2007

ABSTRACT

The axisymmetric problem of the contact interaction of an elastic cover plate with an elastic layer, loaded at infinity with a uniform stretching force, directed parallel to the boundaries of the layer, is considered. The cover plate resists stretching but does not resist bending. The contact shearing stress under the cover plate, the displacement of the points of the cover plate and the deformation distortion coefficient of the elastic layer are determined.

1. Formulation of the problem

Suppose the boundary $z = h_1$ of an elastic layer $0 \le r < \infty$, $0 \le z \le h_1$ on a circle $r \le a$ is reinforced with an elastic cover plate, rigidly attached to the boundary of the elastic layer. The boundary layer $z = 0$ is stress-free. The elastic layer is loaded at infinity by a uniform stretching force p. The cover plate resists stretching but does not resist bending. In this case the strain of the cover plate can be described by an equation[1] with free-edge boundary condition

$$2\theta_2 h_2\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right)u_2(r) = \tau(r), \quad u_2(0) = 0 \tag{1.1}$$

$$\left(\frac{d}{dr} + \frac{\nu_2}{r}\right)u_2(r) \ (r = a) \tag{1.2}$$

Here $\theta_2 = G_2/(1 - \nu_2)$, G_2 and ν_2 are the shear modulus and Poisson's ratio of the cover plate, h_2 is its thickness, $u_2(r)$ is the horizontal displacement of points of the cover plate, and $\tau(r)$ is the contact shearing stress between the lower surface of the cover plate and the upper surface of the elastic layer.

The boundary conditions for the elastic layer have the form

$$\begin{aligned} &\sigma_z^{(1)}(r, h_1), \quad 0 \le r < \infty; \quad \tau_{rz}^{(1)}(r, h_1), \quad a < r < \infty \\ &\sigma_z^{(1)}(r, 0), \quad 0 \le r < \infty; \quad \tau_{rz}^{(1)}(r, 0), \quad a \le r < \infty \\ &u_1(r, h_1) = u_2(r), \quad 0 \le r \le a \end{aligned} \tag{1.3}$$

at infinity

$$\sigma_r^{(1)}(r, z) = p \tag{1.4}$$

while residual stresses vanish. Here $\sigma_r^{(1)}$, $\sigma_z^{(1)}$, $\tau_{rz}^{(1)}$ are the stresses in the elastic layer and u_1 is the horizontal displacement of points of the elastic layer.

☆ *Prikl. Mat. Mekh.* Vol. 72, No. 2, pp. 322–327, 2008.

0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2008.04.003

It is obvious that

$$\tau_{rz}^{(1)}(r, h_1) = \tau(r), \quad 0 \le r \le a \tag{1.5}$$

By searching for a solution of the Lamé equations with boundary conditions (1.3) and (1.4) in the form

$$u_1(r, z) = \frac{(1-\nu_1)pr}{2G_1(1+\nu_1)} + \int_0^\infty U(\gamma, z)\gamma J_1(\gamma r)d\gamma$$

$$w_1(r, z) = -\frac{\nu_1 pz}{2G_1(1+\nu_1)} + \int_0^\infty W(\gamma, z)\gamma J_0(\gamma r)d\gamma \tag{1.6}$$

where G_1 and ν_1 are the shear modulus and Poisson's ratio of the elastic layer, w_1 is the vertical displacement of points of elastic layer, and $J_0(x)$ and $J_1(x)$ are Bessel functions, and using the well-known technique of the Hankel integral transformation,[2] we can reduce problem (1.3), (1.4) to a determination of the function $\tau(r)$ from the integral equation

$$\int_0^a \tau(\rho)\rho d\rho \int_0^\infty L(\gamma h)J_1(\gamma\rho)J_1(\gamma r)d\gamma = -\frac{pr}{2(1+\nu_1)} + \theta_1 u_2(r)$$

$$0 \le r \le a, \quad \theta_1 = \frac{G_1}{1-\nu_1}, \quad L(u) = \frac{\operatorname{sh}2u - 2u}{2(\operatorname{sh}^2 u - u^2)} \tag{1.7}$$

Thus, to determine the functions $\tau(r)$ and $u_2(r)$ we must simultaneously solve differential Eq. (1.1) with boundary condition (1.2) and integral Eq. (1.7).

We will change to dimensionless variables and notation by the formulae

$$r' = \frac{r}{a}, \quad \rho' = \frac{\rho}{a}, \quad \gamma' = a\gamma, \quad \tau(r') = \frac{2}{p}(1+\nu_1)\tau(ar')$$

$$u_2(r') = \frac{2}{pa}(1+\nu_1)\theta_1 u_2(ar')$$

Henceforth omitting the prime, we obtain

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(ru_2(r))\right) = k\tau(r), \quad \left(\frac{d}{dr} + \frac{\nu_2}{r}\right)u_2(r) = 0 \ (r = 1) \tag{1.8}$$

$$\int_0^1 \tau(\rho)\rho d\rho \int_0^\infty L(\gamma\lambda)J_1(\gamma\rho)J_1(\gamma r)d\gamma = \delta(r), \quad 0 \le r \le 1$$

$$\delta(r) = -r + u_2(r), \quad k = \frac{\theta_1 a}{2\theta_2 h_2}, \quad \lambda = \frac{h_1}{a} \tag{1.9}$$

Problem (1.8), (1.9) contains three dimensionless parameters: k, ν_2 and λ. Integral Eq. (1.9) is equivalent to the following paired integral equation

$$\int_0^\infty T(\gamma)L(\gamma\lambda)J_1(\gamma r)d\gamma = -r + u_2(r), \quad 0 \le r \le 1$$

$$\int_0^\infty T(\gamma)J_1(\gamma r)\gamma d\gamma = 0, \quad 1 \le r < \infty \tag{1.10}$$

The Hankel tranformant $T(\gamma)$ is related to the original $\tau(r)$ by the expressions

$$T(\gamma) = \int_0^1 \tau(\rho)J_1(\gamma\rho)\rho d\rho, \quad \tau(r) = \int_0^\infty T(\gamma)J_1(\gamma r)\gamma d\gamma, \quad 0 \le r \le 1$$

Note that

$$L(u) \sim 1 + O(e^{-2u}) \quad \text{as} \quad u \to \infty, \quad L(u) \sim \frac{1}{u} + O(u) \quad \text{as} \quad u \to 0$$

and the inner integral on the left-hand side of Eq. (1.9) converges.

Making the replacement of variables $y=\gamma\lambda$ in integral Eq. (1.9), we obtain

$$\frac{1}{\lambda}\int_0^1 \tau(\rho)\rho d\rho \int_0^\infty L(y)J_1\left(\frac{\rho y}{\lambda}\right)J_1\left(\frac{ry}{\lambda}\right)dy = -r+u_2(r), \quad 0\le r\le 1 \tag{1.11}$$

2. Reduction of the axisymmetric problem to an infinite algebraic system

We will construct a system of orthonormalized polynomials, odd in r, such that

$$Q_m(1)Q_n(1)(1-\nu_2)-\int_0^1 \frac{d}{dr}[rQ_m(r)]\frac{d}{dr}[rQ_n(r)]\frac{dr}{r} = \begin{cases}0, & m\ne n\\ 1, & m=n\end{cases}$$

$$\frac{dQ_m(r)}{dr}+\nu_2 Q_m(1) = 0 \ (r=1) \tag{2.1}$$

The first three polynomials, which satisfy conditions (2.1), have the form

$$Q_1(r) = \sqrt{\frac{3+3\nu_2}{14+2\nu_2}}\left(r^3-\frac{3+\nu_2}{1+\nu_2}r\right)$$

$$Q_2(r) = \sqrt{\frac{70+10\nu_2}{17+\nu_2}}\left(r^5-\frac{3}{2}\frac{9+\nu_2}{7+\nu_2}r^3+\frac{1}{2}\frac{11+\nu_2}{7+\nu_2}r\right)$$

$$Q_3(r) = \sqrt{\frac{2975+175\nu_2}{62+2\nu_2}}\left(r^7-2\frac{19+\nu_2}{17+\nu_2}r^5+\frac{6}{5}\frac{21+\nu_2}{17+\nu_2}r^3-\frac{1}{5}\frac{23+\nu_2}{17+\nu_2}r\right) \tag{2.2}$$

Polynomials (2.2) and those following them can be constructed by a Schmidt orthogonalization process,[3] by means of which a closed system of polynomials is obtained satisfying the second condition of (2.1) and belonging to a Hilbert space with norm, which follows from the scalar product (2.1).

The solution of problem (1.8) can now be represented in the form

$$u_2(r) = \sum_{m=1}^{\infty} a_m Q_m(r), \quad a_n = -k\int_0^1 \tau(r)Q_n(r)rdr \tag{2.3}$$

Substituting the function $u_2(r)$ in the form of (2.3) into integral Eq. (1.11), we obtain

$$\frac{1}{\lambda}\int_0^1 \tau(\rho)\rho d\rho \int_0^\infty L(y)J_1\left(\frac{\rho y}{\lambda}\right)J_1\left(\frac{ry}{\lambda}\right)dy = -r+\sum_{m=1}^{\infty} a_m Q_m(r) \tag{2.4}$$

Since the operator on the left-hand side of integral Eq. (2.4) is linear and the system of polynomials Q_k is linearly independent, the following representation holds

$$\tau(r) = \tau_0(r)+\sum_{m=1}^{\infty} a_m\tau_m(r) \tag{2.5}$$

We will represent the kernel $L(y)$ of integral equation (2.4) in the form $L(y)=1-M(y)$ and substitute expression (2.4) into integral Eq. (2.4) (or the first relation of the paired integral Eq. (1.10)), and also into the second relation of paired integral Eq. (1.10). Equating the coefficients for $a_0, a_1, \ldots$, in the relations obtained assuming $a_0=1$, we conclude that these relations decompose into the following system of paired integral equations

$$I(\tau_0) = -r+K(\tau_0), \quad 0\le r\le 1; \quad J(\tau_0) = 0, \quad r>1 \tag{2.6}$$

$$I(\tau_i) = Q_i(r)+K(\tau_i), \quad 0\le r\le 1; \quad J(\tau_i) = 0, \quad r>1, \quad i = 1,2,\ldots \tag{2.7}$$

Here we have introduced the following notation

$$I(\tau) = \int_0^1 \tau(\rho)\rho d\rho \int_0^\infty J_1(\rho\gamma)J_1(r\gamma)d\gamma = \int_0^\infty T(\gamma)J_1(\gamma r)d\gamma$$

$$J(\tau) = \int_0^1 \tau(\rho)\rho d\rho \int_0^\infty J_1(\rho\gamma)J_1(r\gamma)\gamma d\gamma = \int_0^\infty T(\gamma)J_1(\gamma r)\gamma d\gamma$$

$$K(\tau) = \frac{1}{\lambda}\int_0^1 F\left(\frac{r}{\lambda}, \frac{\rho}{\lambda}\right)\tau(\rho)\rho d\rho$$

$$F\left(\frac{r}{\lambda}, \frac{\rho}{\lambda}\right) = \int_0^\infty M(y)J_1\left(\frac{ry}{\lambda}\right)J_1\left(\frac{\rho y}{\lambda}\right)dy \tag{2.8}$$

We will use an expansion of the Bessel functions in a power series. Then

$$F\left(\frac{r}{\lambda}, \frac{\rho}{\lambda}\right) = k_2 r\rho\frac{1}{\lambda^2} + k_4\frac{(r^2+\rho^2)}{2}\frac{1}{\lambda^4} + k_6\frac{(r^4+3r^2\rho^2+\rho^4)r\rho}{12}\frac{1}{\lambda^6} + O\left(\frac{1}{\lambda^8}\right)$$

$$k_n = \frac{1}{2^n}\int_0 M(u)u^n du, \quad n = 2, 4, 6 \tag{2.9}$$

We will seek $\tau_k(r)$ $(k=0,1,2,\ldots)$ in the form of a series

$$\tau_k(r) = \sum_{n=0}^{\infty} \tau_{kn}(r)\frac{1}{\lambda^n} \tag{2.10}$$

Substituting expansion (2.9) and (2.1) into Eqs. (2.6) and (2.7) and equating coefficients of like powers of λ, we obtain the following chain of paired integral equations

$$\begin{aligned}
&I(\tau_{00}) = -r, \quad 0 \le r \le 1; \quad J(\tau_{00}) = 0, \quad r > 1\\
&I(\tau_{i0}) = Q_i(r), \quad 0 \le r \le 1; \quad J(\tau_{i0}) = 0, \quad r > 1\\
&I(\tau_{03}) = k_2 r N_{00}, \quad 0 \le r \le 1; \quad J(\tau_{03}) = 0, \quad r > 1\\
&I(\tau_{i3}) = k_2 r N_{i3}, \quad 0 \le r \le 1; \quad J(\tau_{i3}) = 0, \quad r > 1\\
&I(\tau_{04}) = k_2 r N_{04}, \quad 0 \le r \le 1; \quad J(\tau_{04}) = 0, \quad r > 1\\
&I(\tau_{i4}) = k_2 r N_{i4}, \quad 0 \le r \le 1; \quad J(\tau_{i4}) = 0, \quad r > 1\\
&I(\tau_{05}) = -1/2k_4(r^3 N_{00} + rM_{00}), \quad 0 \le r \le 1; \quad J(\tau_{05}) = 0, \quad r > 1\\
&I(\tau_{i5}) = -1/2k_4(r^3 N_{i0} + rM_{i0}), \quad 0 \le r \le 1; \quad J(\tau_{i5}) = 0, \quad r > 1
\end{aligned} \tag{2.11}$$

Here we have introduced the notation

$$N_{ij} = \int_0^1 \tau_{ij}(\rho)\rho^2 d\rho, \quad M_{ij} = \int_0^1 \tau_{ij}(\rho)\rho^4 d\rho$$

Note also that

$$\tau_{0l} = \tau_{il} = 0, \quad l = 1, 2$$

To find $\tau_{\rm mn}$ (m, $n=0, 1, 2, \ldots$) it remains to solve the corresponding paired integral equations for $\tau_{\rm mn}$.

We will write the paired integral equation in the general form

$$\int_0^\infty T(\gamma)J_1(\gamma r)d\gamma = f(r), \quad 0 \le r \le 1$$

$$\int_0^\infty T(\gamma)J_1(\gamma r)\gamma d\gamma = 0, \quad 1 \le r < \infty \tag{2.12}$$

where the Hankel transformant $T(\gamma)$ is related to the original $\tau(r)$ by the expressions

$$T(\gamma) = \int_0^1 \tau(\rho) J_1(\gamma\rho)\rho d\rho, \quad \tau(r) = \int_0^\infty T(\gamma) J_1(\gamma r)\gamma d\gamma, \quad 0 \le r \le 1 \tag{2.13}$$

To solve Eq. (2.12) we will use the following formulae[4]

$$\int_0^t \frac{J_1(\gamma r)}{\sqrt{t^2 - r^2}} dr = \frac{1 - \cos\gamma t}{\gamma t}, \quad \int_t^\infty \frac{J_1(\gamma r)}{\sqrt{r^2 - t^2}} dr = \frac{\sin\gamma t}{\gamma t} \tag{2.14}$$

We multiply the first relation of (2.12) by $(t^2 - r^2)^{-1/2}$ and integrate with respect to r from 0 to t, and we multiply the second relation of (2.12) by $(r^2 - t^2)^{-1/2}$ and integrate with respect to r from t to ∞. Changing in the order of integration and using formulae (2.14), we obtain

$$\int_0^\infty T(\gamma) \frac{1 - \cos\gamma t}{\gamma t} d\gamma = \int_0^t \frac{f(r)}{\sqrt{t^2 - r^2}} dr, \quad 0 \le t \le 1$$

$$\int_0^\infty T(\gamma) \sin\gamma t d\gamma = 0, \quad 1 \le t < \infty \tag{2.15}$$

We now multiply by t and then differentiate the first relation of (2.15) with respect to t. As a result we have

$$\int_0^\infty T(\gamma) \sin\gamma t d\gamma = \begin{cases} p(t) \ (0 \le t \le 1) \\ 0 \ (1 \le t \le \infty) \end{cases}$$

$$p(t) = \frac{d}{dt} t \int_0^t \frac{f(r) dr}{\sqrt{t^2 - r^2}} \tag{2.16}$$

From the first formula of (2.16) we obtain, using a Fourier sine-integral transformation,

$$T(\gamma) = \frac{2}{\pi} \int_0^1 p(t) \sin\gamma t dt \tag{2.17}$$

Finally, substituting expression (2.17) into the second relation of (2.13), changing the order of integration and using a discontinuous Sonin integral[4]

$$\int_0^\infty \sin\gamma t J_0(\gamma r) d\gamma = \begin{cases} (t^2 - r^2)^{-1/2}, & 0 < r < t \\ 0, \quad 0 < t < r \end{cases}$$

we obtain

$$\tau(r) = -\frac{2}{\pi} \frac{d}{dr} \int_r^1 \frac{p(t) dt}{\sqrt{t^2 - r^2}} = -\frac{2}{\pi} \frac{d}{dr} \int_r^1 \frac{d}{dt} \left(t \int_0^t \frac{f(r) dr}{\sqrt{t^2 - r^2}} \right) \frac{dt}{\sqrt{t^2 - r^2}} \tag{2.18}$$

We have taken Eq. (2.17) into account here.

Hence, by solving the paired integral Eq. (2.11), we obtain τ_{mn} (m, $n = 0, 1, 2, \ldots$) and $\tau(r)$:

$$\tau(r) = \tau_0(r) + \sum_{m=1}^\infty a_m \tau_m(r) = \sum_{n=0}^\infty \tau_{0n}(r) \frac{1}{\lambda^n} + \sum_{m=1}^\infty a_m \sum_{n=0}^\infty \tau_{mn}(r) \frac{1}{\lambda^n} \tag{2.19}$$

Substituting the expression obtained for $\tau(r)$ into the second formula of (2.3), we obtain an infinite algebraic system in the coefficients a_m of expression (2.3)

$$a_m = -k \int_0^1 \left(\sum_{n=0}^\infty \frac{1}{\lambda^n} \left(\tau_{0n}(r) + \sum_{m=1}^\infty a_m \tau_{mn}(r) \right) \right) Q_n(r) r dr \tag{2.20}$$

After finding an approximation solution of the algebraic system (2.20) (for example, by the reduction method[5]) the displacements of the points of the cover plate can be found from the first formula of (2.3), the contact shearing stress can be found using formula (2.19), while the coefficient which indicates by what factor the presence of the cover plate reduces the strain of the layer without the cover plate is found from the formula

$$\kappa = \left(2\int_0^1 u_2'(r)rdr\right)^{-1}$$

For $\nu_2 = 0.3$ and different values of the parameters k and λ, we have

k	4	2	1	
κ	3.38	2.22	1.62	for $\lambda = 2$
κ	3.35	2.19	1.58	for $\lambda = 8$

Acknowledgement

This research was financed by the Russian Foundation for Basic Research (08-01-00003, 06-01-00022, 06-08-01595, 07-08-00730).

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Translated by R.C.G.